Quantum Physics, Course KFY/7KVAF

WS 2020/2021

Seminar 2: Properties of linear operators

1. Calculate commutator
a) $[\hat{A}\hat{B},\hat{C}]$ of three operators $\hat{A},\hat{B},\hat{C}$ using $[\hat{A},\hat{C}]$ a $[\hat{B},\hat{C}]$,
b) $[\frac{\mathrm{d}}{\mathrm{d}x},x]$,
c) of position $\hat{x}$ and momentum $\hat{p}$ operators,
d) $[\hat{x}-\hat{p},\hat{p}-\hat{x}]$,
e) $[\hat{x}\hat{p},\hat{x}]$,
f) $[\frac{\partial}{\partial x},f(x,y,z)]$, where f is complex function,
g) $[\hat{x},\Delta]$.

2. Which of the following $\hat{A}$ operators are linear ($u \equiv u(x)$ is complex function)?
a) $\hat{A}u = \lambda u$, where λ is complex constant,
b) $\hat{A}u = u^*$,
c) $\hat{A}u = u^2$,
d) $\hat{A}u = \frac{\mathrm{d}u}{\mathrm{d}x}$,
e) $\hat{A}u = \frac{1}{u}$,
f) $\hat{A}u = \frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 3u^2$.

3. Find adjoint operators to operators a) – e) and check possible self-adjoint operators (Hermitian operator):
a) $\hat{A} = x$,
b) $\hat{A} = \frac{\mathrm{d}}{\mathrm{d}x}$,
c) $\hat{A} = f(x)$, $f(x)$ is complex function,
d) $\hat{A} = f(x)$, $f(x)$ is real function,
e) $\hat{A} = c$, c is complex number.

4. Check if the following operators are Hermitian
a) the momentum operator $\hat{p} = -ih\nabla$,
b) the Laplace operator Δ.

5. Find the eigenvalues and corresponding eigenvectors of the following matrices:
a) $\begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix}$
b) $\begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & 2 \end{pmatrix}$
c) two-level Hamiltonian $\hat{H} = \begin{pmatrix} E_0 & A \\ A & E_0 \end{pmatrix}$, $E_0, A \in \mathbb{R}$.

6. Find the eigenvalues and corresponding eigenvectors of the following operators:
a) $\frac{\mathrm{d}}{\mathrm{d}x}$,
b) i($\frac{\mathrm{d}}{\mathrm{d}x}$),
c) $x + \frac{\mathrm{d}}{\mathrm{d}x}$,
d) -i($\frac{\mathrm{d}}{\mathrm{d}\varphi}$), where φ is rotation by angle around the axis z (spherical coordinate).

7. Match the eigenfunctions in right column to their operators in left column. What is the eigenvalue for each eigenfunction?

i.	$(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x\frac{\mathrm{d}}{\mathrm{d}x}$	a.	$4x^4 - 12x^2 + 3$
ii.	$\frac{\mathrm{d}^2}{\mathrm{d}x^2}$	b.	$5x^4$
iii.	$x\frac{\mathrm{d}}{\mathrm{d}x}$	c.	$e^{3x} + e^{-3x}$
iv.	$\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}}{\mathrm{d}x}$	d.	$x^2 - 4x + 2$
v.	$x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (1-x)\frac{\mathrm{d}}{\mathrm{d}x}$	e.	$4x^3 - 3x$