Quantum Physics, Course KFY/7KVAF
WS 2022/2023
Theme 1: Wavefunctions, vector spaces, linear operators

1. Which of the following functions can be wave functions corresponding to the stationary state of a particle in an interval $x \in (-\infty, \infty)$?
a) $\psi(x) = x$ for $x \geq 0$ and $\psi(x) = x$ for $x < 0$,
b) $\psi(x) = x^2$,
c) $\psi(x) = e^{-|x|}$,
d) $\psi(x) = e^{-x}$,
e) $\psi(x) = \cos x$,
f) $\psi(x) = \sin |x|$,
g) $\psi(x) = e^{-x^2}$,
h) $\psi(x) = 1$ for $-1 \leq x \leq 1$ and $\psi(x) = 0$ for $x < -1$ and $x > 1$,
i) $\psi(x) = x(a-x)$ for $0 \leq x \leq a$ and $\psi(x) = 0$ for $x < 0$ and $x > a$.

2. Can the following functions describe an identical quantum state?
a) $\psi(x)$ and $c\psi(x)$, where c is real constant,
b) $\psi(x)$ and $e^{if(x)}\psi(x)$, where $f(x)$ is real function.

3. Check the linear dependency/independency of vectors and functions
a) $\vec{a} = (3, 2, 7)$; $\vec{b} = (1, 1, 1)$; $\vec{c} = (2, 0, 3)$,
b) $\vec{a} = (3, 2, 0)$; $\vec{b} = (1, 1, 1)$; $\vec{c} = (5, 4, 2)$,
c) $\vec{a} = (1.5, 2)$ a $\vec{b} = (2.5, 3)$ and if indenpendency is proved, use both vectors for expression of third one $\vec{x} = (4, 1)$ as linear combination,
d) $\vec{a} = (3, 4, 5)$; $\vec{b} = (-6, 7, 0)$; $\vec{c} = (8, -9, 1)$ and use these vectors for expression of fourth one $\vec{x} = (23, -19, 6)$,
e) $\psi_1(x) = x^2$; $\psi_2(x) = x$; $\psi_3(x) = \frac{1}{x}$; $\psi_4(x) = 1$,
f) $\psi_1(x) = 1 + x$; $\psi_2(x) = x + x^2$; $\psi_3(x) = x^2 + x^3$; $\psi_4(x) = x^3 + 1$.

4. Check the scalar product axioms for the following operations
a) $(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{N} x_i y_i$ defined in $\mathbb{R}^N$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$,
b) $(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{N} x_i y_i^*$ defined in $\mathbb{C}^N$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^N$,
c) $\langle \psi, \varphi \rangle = \int_a^b \psi(x)\varphi(x)\mathrm{d}x$ for all functions $\psi(x), \varphi(x) \in L^{\mathbb{R}}(a, b)$,
d) $\langle \psi, \varphi \rangle = \int_a^b \psi^*(x)\varphi(x)\mathrm{d}x$ for all functions $\psi(x), \varphi(x) \in L^{\mathbb{C}}(a, b)$.

5. Proof the orthogonality of the following systems of functions in Hilbert space L_2
a) $\{e^{\mathrm{i}nx}\}_{n=-\infty}^{\infty}$ in $[0,2\pi]$,
b) $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), ...\}$ in $(-\pi,\pi)$,
c) Legendre polynomials in (-1,1) given by $L_0(x) = 1$, $L_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2 - 1)^n$ (try to express first four and check selected combinations).

6. Replace the following classical mechanical expressions with their corresponding quantum-mechanical operators
a) kinetic energy $T = \frac{1}{2}mv^2$ in three-dimensional space,
b) $\boldsymbol{p} = m\boldsymbol{v}$, a three-dimensional cartesian vector,
c) y-component of angular momentum $L_y = zp_x - xp_z$.

7. Calculate
a) $(\hat{A} - \hat{B})(\hat{A} + \hat{B})$,
b) $\hat{A}^2$ corresponding to operator $\hat{A} = \frac{\mathrm{d}}{\mathrm{d}x} + x$,
c) $\hat{A}^3$ corresponding to operator $\hat{A} = \frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{x}$,
d) and compare operators $(x\frac{\mathrm{d}}{\mathrm{d}x})^2$ and $(\frac{\mathrm{d}}{\mathrm{d}x}x)^2$.